ON EQUALITY OF GENERALIZED INVERSES USED IN SOLVING NORMAL EQUATIONS

S. R. Searle and E. C. Townsend
Biometrics Unit, Cornell University, Ithaca, New York

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Abstract

Statement and proof are given of necessary and sufficient conditions for the equality, established by John (1964), relating two common forms of a generalized inverse matrix used in solving normal equations in the general linear model. The sufficient condition establishes that the relationship holds whenever side conditions that are independent of the design matrix are added to the normal equations.

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1. Summary

Statement and proof are given of necessary and sufficient conditions for the equality, established by John (1964), relating two common forms of a generalized inverse matrix used in solving normal equations in the general linear model. The sufficient condition establishes that the relationship holds whenever side conditions that are independent of the design matrix are added to the normal equations.

2. A Theorem

Normal equations derived by least squares for the familiar linear model $y = X\beta + e$ are $X'X\beta = X'y$. When X has order $n \times p$ and rank $p - m$ ($n > p$, $m > 0$), these equations can be solved by using side conditions $H\beta = 0$ and the inverse

$$\begin{bmatrix} X'X & H \\ H' & 0 \end{bmatrix}^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} . \qquad \text{- - - (1)}$$

John (1964) shows that B_{11} is a Rao (1962) generalized inverse of $X'X$ and that

$$B_{11} = (X'X + H'H)^{-1} - D(D'H'HD)^{-1}D', \qquad \text{- - - (2)}$$

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where D is $p \times m$, of rank m, with $XD = 0$. By (2), $(X'X + H'H)^{-1}$ is also a generalized inverse of $X'X$. In establishing these results John (1964) requires that H, of order $m \times p$, be "such that HD has rank m", i.e. that HD, which is $m \times m$, be of full rank. This limitation on H appears restrictive — we show that it is not.

The normal equations $X'X\beta = X'y$ subject to $H\beta = 0$ can be written as

$$\begin{bmatrix} X'X & H' \\ H & 0 \end{bmatrix} \begin{bmatrix} \beta \\ 0 \end{bmatrix} = \begin{bmatrix} X'y \\ 0 \end{bmatrix} \qquad \text{--- (3)}$$

and can be solved using (1) only if that inverse exists. This it will do provided H is $m \times p$, is of rank m and has its rows linearly independent of those of X. The equivalence of these conditions to that required by John (1964), that HD be of full rank, is represented in the following theorem.

Theorem. Given X of order $n \times p$ and rank $p - m$ $(n > p,\ m > 0)$, and corresponding matrices D of order $p \times m$ and rank m, such that $XD = 0$, then a necessary and sufficient condition for H, of order $m \times p$, to be such that HD has full rank, is that H be of rank m and have its rows linearly independent of those of X.

John (1964) demands that HD have full rank and utilizes this property in the proofs relating to B_{11}, and the necessity condition of the theorem, easily proved, is clearly implied. However, the sufficient condition is the all-important one, because the equations (3) will always be set up not by finding and using an H for which HD is full rank, but by using any H that has rank m and whose rows are linearly independent of those of X. In order to use (2) after solving (3) by means of (1), it is therefore necessary to prove the sufficient condition of the theorem, and especially so since, as shall be seen, its proof is not a mere reversal of the proof of the necessary condition.

3. Proof of theorem

Necessity. If HD, square of order m, is of full rank, then clearly the rank of $H_{m \times p}$ must be m. Assuming that the rows of H are linearly dependent on those of X, then there exists some matrix K for which H = KX. Post-multiplying by D gives HD = KXD and hence HD = 0 which contradicts the full rank property of HD. Therefore the rows of H are linearly independent of those of X, and thus the necessity condition is proven.

Sufficiency. If $H_{m \times p}$ has rank m and its rows are linearly independent of the rows of X, then the rank of the partitioned matrix $\begin{bmatrix} X \\ H \end{bmatrix}$ is p. Therefore this matrix has a left inverse [e.g., Searle (1966) Section 5.13], (U V) say, and so

$$UX + VH = I \ .$$

Therefore $$UXD + VHD = D \ ,$$

leading to $$VHD = D \ .$$

But $D_{p \times m}$ has rank m, and so it too has a left inverse, E say, so that

$$EVHD = I_{m \times m}$$

$$\therefore \qquad \text{rank}(HD) \geq m \ ,$$

and since HD has order m it must be of full rank.

4. Penrose's conditions

In accord with the definition of Rao (1962) B_{11} is described as a generalized inverse of X'X because $X'XB_{11}X'X = X'X$, thus satisfying the first of the four Penrose (1955) conditions. In passing, note that it also satisfies the second condition, because $B_{11}X'XB_{11} = B_{11}$, this being readily observed from $X'XB_{11} = I - H'(D'H')^{-1}D'$ and $HB_{11} = 0$ given by John (1964). And from (2) it is clear that $(X'X + H'H)^{-1}$ satisfies the first two conditions also. But neither this nor B_{11} satisfy the other two Penrose conditions: e.g. neither $B_{11}X'X$ nor $X'XB_{11}$ are symmetric.

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